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An intersection theory for hypergeometric functions

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Acknowledgement.

Thank you very much for inviting me to give a talk in this symposium. This talk is based on a joint work with Michitake Kita in Kanazawa University. I would like to thank Keiji Matsumoto for giving us the opportunity of this collaboration.

1. What are HGF's ?

(1.1) Classical HGF's.

In this talk, I am talking about hypergeometric functions (HGF's). What are HGF's ? The most classical ones ares the Gauss HGF's; they are solutions of the Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2f}{dz^2}+\{c-(a+b+1)z\}\frac{df}{dz}-abf=0 \quad \text{on} \quad \mathbf{P}^1.$$

Late in the nineteenth century, P. Appell [**Ap**] and G. Lauricella [**La**] introduced HGF's of several variables.

P. Appell (1880) —— 2 variables, F_1, F_2, F_3, F_4,
G. Lauricella (1893) — n variables F_D, F_A, F_B, F_C (a century ago!).

The HGF's have been considered as one of the *most important special functions*, because they have quite many applications to various fields in mathematics as well as in mathematical physics.

(1.2) Aomoto-Gelfand HGF's.

In 1986, after a series of pioneering works by K. Aomoto, I.M. Gel'fand [Ge] defined a class of HGF's of several variables. In fact, Aomoto [Ao] gave essentially the same definition in 1975. Their definitions are quite natural, simple and beautiful. Recently, mathematics related to Grassmannian manifolds has been quite active. The Aomoto-Gel'fand HGF's are an example of such a *Grassmannian mathematics*.

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(1.3) Fibrations.

Let $\overline{M} = \overline{M}(m+1, n+1)$ be the set of all $(m+1) \times (n+1)$-complex matrices of *full rank*:

$$\begin{aligned}\overline{M} &= \overline{M}(m+1, n+1) \\ &:= \{z\ ;\ (m+1) \times (n+1)\text{-complex matrix of full rank}\},\end{aligned}$$

with $m > n$, $M = M(m+1, n+1)$ the set of all matrices *in general position*:

$$\begin{aligned}M &= M(m+1, n+1) \\ &:= \{z \in \overline{M}\ ;\ z \text{ is in general position}\},\end{aligned}$$

where z is said to be in general position if *all* $(n+1)$*-minors of* z *do not vanish.*

We regard $\mathbf{C}^{m+1}$ and $\mathbf{C}^{n+1}$ as a *column vector* space with coordinates

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{pmatrix}, \qquad u = \begin{pmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix},$$

respectively. These coordinates are regarded also as homogeneous coordinates of the projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$, respectively.

Consider a fibration $\pi : \overline{E} \to \overline{M}$ defined by

$$\overline{E} = \overline{E}(m+1, n+1) := \{(z, u) \in \overline{M} \times \mathbf{P}^n\ ;\ \prod_{i=0}^{m} x_i(zu) \neq 0\}$$

where $\pi : \overline{E} \to \overline{M}$ is the projection into the first component. Let

E_z : the fiber of $\overline{E}$ over $z \in \overline{M}$ ("bar" is omitted).

We put

$E := \overline{E}|_M$: restriction of the base space of $\overline{E}$ to M.

LEMMA 1.3.1. *$\pi : E \to M$ is a topological fiber bundle i.e. topologically locally trivial.*

(1.4) Local systems.

Let A be an affine parameter space defined by

$$A = A(m+1, n+1) := \{\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{pmatrix} \in \mathbf{C}^{m+1} \; ; \; \sum_{i=0}^{m} \alpha_i = -(n+1)\}.$$

For any $\alpha \in A$, we consider a multi-valued holomorphic section f of $\mathcal{O}_{\overline{E}/\overline{M}}(-n-1)$ defined by

$$f = f(z,u) = f(z,u;\alpha) := \prod_{i=0}^{m} z_i(zu)^{\alpha_i}.$$

Since f is *homogeneous of degree* $-n-1$ with respect to u, f is indeed a "section" of $\mathcal{O}_{\overline{E}/\overline{M}}(-n-1)$. Let

$\mathcal{L} = \mathcal{L}_\alpha$: local system on $\overline{E}$ over the field $\mathbf{C}$ such that
each branch of f determines a horizontal local section of $\mathcal{L}$,
$\mathcal{L}^\vee$: dual local system of $\mathcal{L}$ on $\overline{E}$,
$\mathcal{L}_z := \mathcal{L}|_{E_z}$: restriction of $\mathcal{L}$ to each fiber E_z,
$\mathcal{L}_z^\vee := \mathcal{L}^\vee|_{E_z}$: restriction of $\mathcal{L}^\vee$ to each fiber E_z.

(1.5) Twisted (co-)homology.

Let

$\mathcal{H}^q = \mathcal{H}^q(m+1, n+1; \alpha)$
$:= \mathcal{H}^q(E, \mathcal{L})$
: q-th twisted cohomology of $(E, \mathcal{L})$ along the fibers of $\pi : E \to M$,
$\mathcal{H}_q^\vee = \mathcal{H}_q^\vee(m+1, n+1; \alpha)$
$:= \mathcal{H}_q(E, \mathcal{L}^\vee)$
: q-th twisted homology of $(E, \mathcal{L}^\vee)$ along the fibers of $\pi : E \to M$.

Namely,

$$\mathcal{H}^q = \bigcup_{z \in M} H^q(E_z, \mathcal{L}_z), \qquad \mathcal{H}_q^\vee = \bigcup_{z \in M} H_q(E_z, \mathcal{L}_z^\vee).$$

There are natural projections

$$\pi : \mathcal{H}^q \to M \qquad \pi : \mathcal{H}_q^\vee \to M.$$

By Lemma 1.3.1, we have the following:

LEMMA 1.5.1. *$\pi : \mathcal{H}^q \to M$ and $\pi : \mathcal{H}_q^\vee \to M$ admit natural structures of local system on M.*

(1.6) Hypergeometric functions (HGF's).

We denote by

$$\mathcal{H}_q^\vee \otimes \mathcal{H}^q \to \mathbf{C}_M, \quad (c, \varphi) \mapsto \int_c \varphi$$

the *fiberwise pairing* of the homology and the cohomology, where $\mathbf{C}_M$ is the constant system on M with fiber $\mathbf{C}$.

Let $du := du_0 \wedge du_1 \wedge \cdots \wedge du_n$ be the standard volume form on $\mathbf{C}^{n+1}$. The interior product of du by the Euler vector field

$$e = \sum_{i=0}^{n} u_i \frac{\partial}{\partial u_i} \quad : \text{Euler vector field}$$

defines an $\mathcal{O}_{\mathbf{P}^n}(n+1)$-valued n-form

$$\omega = \iota_e du \qquad \text{on} \quad \mathbf{P}^n,$$

Pulling back this form to $\overline{E}$, we obtain an *$\mathcal{O}_{\overline{E}/\overline{M}}(n+1)$-valued n-form along the fibers of $\pi : \overline{E} \to \overline{M}$*. We denote it also by ω. Put

$$\varphi(z) = \varphi(z; \alpha) := f(z, u; \alpha)\omega.$$

This n-form along the fibers determines an element of $H^n(E_z, \mathcal{L}_z)$ at each $z \in M$.

DEFINITION 1.6.1: A *hypergeometric function* of type $(m+1, n+1; \alpha)$ is a (germ of) function of the form

$$F(z; \alpha) := \int_{c(z)} \varphi(z),$$

where $c(z)$ is a horizontal local section of $\pi : \mathcal{H}_n^\vee \to M$.

LEMMA 1.6.2. *The HGF $F(z; \alpha)$ is (continued to) a multi-valued holomorphic function on M with regular singularities along $\overline{M} \setminus M$.*

2. Some properties of HGF's.

(2.1) Relation with classical HGF's.

Our HGF's are functions of matrix arguments. By a reduction of arguments, our HGF's *of special type* reduce to the classical HGF's.

Lemma 2.1.1.

(1) *The* $(4,2)$-*type reduces to the Gauss HGF.*
(2) *The* $(m+1,2)$-*type reduces to the Lauricella HGF* F_D *of* $(m-2)$-*variables.*

The Lauricella hypergeometric series of n-variables is defined by

$$\begin{aligned} F_D &= F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &= \sum \frac{(a, m_1 + \cdots + m_n)(b_1, m_1) \cdots (b_n, m_n)}{(c, m_1 + \cdots + m_n) m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \end{aligned}$$

where the sum is taken over all nonnegative integers $m_1, \ldots, m_n$ and $(a, m) := a(a+1) \cdots (a+m-1)$. If $\Re(b_i)$, $(i = 1, \ldots, n)$ and $\Re(c-b)$ are positive, then F_D admits the following *Euler integral* representation:

$$F_D = \text{const.} \int \cdots \int_\Delta \prod_i u_i^{b_i - 1} (1 - \sum_i u_i)^{c-b-1} (1 - \sum_i t_i x_i)^{-a} dt_1 \cdots dt_n,$$

where

$$\begin{aligned} b &:= \sum_i b_i, \\ \text{const.} &:= \frac{\Gamma(c)}{\Gamma(b_1) \cdots \Gamma(b_n) \Gamma(c-b)}, \\ \Delta &:= \{(u_1, \ldots, u_n) \in \mathbf{R}^n \ : \ u_i \geq 0, \sum_i u_i \leq 1\}. \end{aligned}$$

The HGF's admit group actions and the reduction of arguments is made by using these group actions.

(2.2) Group actions.

Groups we are concerned are:

$$\begin{aligned} GL &= GL(n+1) \ : \ \text{complex general group}, \\ H &= H(m+1) \ : \ \text{complex } (m+1)\text{-torus} \\ &:= \{h = \begin{pmatrix} h_0 & & & \\ & h_1 & & \\ & & \ddots & \\ & & & h_m \end{pmatrix} \ ; \ h_i \in \mathbf{C}^\times\} \end{aligned}$$

Actions are given by

$$\begin{aligned} E \times GL \to E, &\qquad ((z,u), g) \mapsto (zg, g^{-1}u), \\ H \times E \to E, &\qquad (h, (z,u)) \mapsto (hz, u). \end{aligned}$$

These actions induce the following *group covariance* of the HGF's:

LEMMA 2.2.1.

(1) $F(zg;\alpha) = (det g)^{-1}F(z;\alpha)$, $\quad (g \in GL)$,
(2) $F(hz;\alpha) = h^{\alpha}F(z;\alpha)$, $\quad (g \in H)$,

where $h^{\alpha} = h_0^{\alpha_0}h_1^{\alpha_1}\cdots h_m^{\alpha_m}$.

Put

$$\overline{G} = \overline{G}(m+1, n+1) := \overline{M}/GL, \qquad G = M/GL.$$

Then $\overline{G}$ is the Grassmannian manifold of $(m+1, n+1)$-type and G is a Zariski open subset of $\overline{G}$.

REMARK 2.2.2: (1) The GL-covariance (1) implies that the HGF's are multi-valued holomorphic sections of the *anti-determinant line bundle* over G.

(2) As for the H-covariance (2), we note that

$H \setminus \overline{M}$: configuration space of $(m+1)$-hyperplanes in $\mathbf{P}^n$,
$H \setminus \overline{M}/GL$: configurations of $(m+1)$-hyperplanes in $\mathbf{P}^n$ up to $\mathrm{Aut}(\mathbf{P}^n)$.

(2.3) Gel'fand system.

LEMMA 2.3.1. *The HGF $F = F(z;\alpha)$ satisfies the following system of PDE's:*

$$\begin{cases} \sum_{k=0}^{m} z_{ki}F_{kj} = -\delta_{ij}F & (0 \le i, j \le n) \\ \sum_{i=0}^{n} z_{ki}F_{ki} = \alpha_k F & (0 \le k \le m) \\ F_{ki;hj} = F_{hi;kj} & (0 \le i, j \le n, 0 \le k, h \le m) \end{cases}$$

where

$$F_{kj} := \frac{\partial F}{\partial z_{kj}}, \qquad F_{ki;hj} := \frac{\partial^2 F}{\partial z_{ki}\partial z_{hj}}.$$

This system, called the *Gel'fand system*, is a regular holonomic system.

3. Exterior product structure.

(3.1) Segre embedding.

The *Segre embedding*:

$$M(m+1, 2) \xrightarrow{\text{Segre}} M(m+1, n+1)$$

is defined by

$$w = \begin{pmatrix} \cdots & \cdots \\ w_{i0} & w_{i1} \\ \cdots & \cdots \end{pmatrix} \mapsto z = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{i0}^n & w_{i0}^{n-1} w_{i1} & w_{i0}^{n-2} w_{i1}^2 & \cdots & w_{i1}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

This is indeed an embedding, because we have the formula:

$$z \begin{pmatrix} i_0 \\ i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_n \end{pmatrix} = \text{nonzero const.} \prod_{p<q} w \begin{pmatrix} i_p \\ i_q \end{pmatrix},$$

where the left-hand side is the $(n+1)$-minor of z determined by the i_0-th, i_1-th, $\ldots$, i_n-th columns of z, the right-hand side being defined in a similar manner. We would like to consider the pull-back of the local systems $\mathcal{H}^q(m+1,n+1;\alpha)$ and $\mathcal{H}_q^\vee(m+1,n+1;\alpha)$ on $M(m+1,n+1)$ by the Segre embedding:

$$\text{Segre}^* \mathcal{H}^q(m+1,n+1;\alpha), \qquad \text{Segre}^* \mathcal{H}_q^\vee(m+1,n+1;\alpha).$$

They are local systems on $M(m+1,2)$. Are there any relation between them and the HGF's of type $(m+1,2)$?

(3.2) Reduction of the base ring.

Until now, $\mathcal{L} = \mathcal{L}_\alpha$ $(\alpha \in A)$ has been considered as a local system over the complex number field $\mathbf{C}$.

$\mathcal{L} = \mathcal{L}_a$ $(a \in A)$: defined over $\mathbf{C}$ — until now.

We put

$$c_i = \exp(2\pi\sqrt{-1}\alpha_i) \in \mathbf{C}^\times, \qquad (i = 0,1,\ldots,m).$$

Since $\sum \alpha_i = 0$, we have

$$(*) \qquad c_0 c_1 \cdots c_m = 1.$$

Now let R be a *subring* of $\mathbf{C}$ such that

$$\mathbf{Q}[c_0^{\pm 1}, c_1^{\pm 1}, \ldots, c_m^{\pm 1}] \subseteq R \subseteq \mathbf{C}.$$

Then the local system $\mathcal{L} = \mathcal{L}_\alpha$ can be defined over the ring R. So, from now on, we assume that $\mathcal{L}$ is defined over R.

$\mathcal{L} = \mathcal{L}_\alpha$: defined over R — from now on.

This *reduction of the base ring* will enable us to study HGF's more precisely. This is especially the case when the parameter $\alpha \in A$ takes a special value in a number-theoretical sense.

(3.3) Exterior product structure.

Let I_R be the ideal of R generated by $1 - c_0, 1 - c_1, \ldots, 1 - c_m$:

$$I_R := \sum_{i=0}^{m} R(1 - c_i).$$

REMARK 3.3.1: In fact, I_R is generated by $1 - c_1, 1 - c_2, \ldots, 1 - c_m$, because $(*)$ implies

$$c_0 - 1 = \sum_{i=1}^{m} \frac{1 - c_i}{c_1 c_2 \cdots c_i}.$$

The following theorem is the main result of this talk:

THEOREM 3.3.2. *Assume $I_R = R$.*

(1) There exist canonical isomorphisms of R-modules:

$$\mathit{Segre}^* \mathcal{H}^q(m+1, n+1; \alpha) \simeq \begin{cases} \overset{n}{\wedge} \mathcal{H}^1(m+1, 2; \alpha) & (q = n) \\ 0 & (q \neq n) \end{cases}$$

$$\mathit{Segre}^* \mathcal{H}^\vee_q(m+1, n+1; \alpha) \simeq \begin{cases} \overset{n}{\wedge} \mathcal{H}^\vee_1(m+1, 2; \alpha) & (q = n) \\ 0 & (q \neq n) \end{cases}$$

(2) Let

$H^q(m+1, 2; \alpha)$: *any fiber of* $\pi : \mathcal{H}^q(m+1, 2; \alpha) \to M(m+1, 2)$,
$H^\vee_q(m+1, 2; \alpha)$: *any fiber of* $\pi : \mathcal{H}^\vee_q(m+1, 2; \alpha) \to M(m+1, 2)$.

Then we have

$$H^q(m+1, 2; \alpha) = 0 = H^\vee_q(m+1, 2; \alpha) \qquad (q \neq 1),$$
$$H^1(m+1, 2; \alpha) \simeq V \simeq H^\vee_1(m+1, 2; \alpha) \qquad (\simeq \ : \textit{not canonical}),$$

where V is an R-module defined by

$$V = \{r = \begin{pmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_m \end{pmatrix} \in R^m \; ; \; \sum_{i=1}^{m} r_i(1-c_i) = 0\}.$$

REMARK 3.3.3: (1) Recall that

$$\pi : \mathcal{H}^q(m+1, n+1; \alpha) \to M(m+1, n+1),$$
$$\pi : \mathcal{H}_q^\vee(m+1, n+1; \alpha) \to M(m+1, n+1)$$

are local systems of R-modules on $M(m+1,n+1)$. Hence, by "analytic continuation", Theorem 3.3.2 determines the R-module structure of the fiber over any point $z \in M(n+1, m+1)$ of these local systems.

(2) If $\mathcal{L}$ is *trivial*, then there exists no such ring R that satisfies the assumption of Theorem 3.3.2.

(3) If $\mathcal{L}$ is *not trivial*, i.e. there exists an i $(1 \leq i \leq m)$ such that $c_i \neq 1$, then the ring

$$R := \mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, \ldots, c_m^{\pm 1}, \frac{1}{1-c_i}]$$

satisfies the assumption of Theorem 3.3.2. In this case, V is a free R-module of rank $m-1$, and hence

$\mathcal{H}^n(m+1, n+1; \alpha)$ *and* $\mathcal{H}_n^\vee(m+1, n+1; \alpha)$ *are local systems of free R-modules of rank*

$$\binom{m-1}{n}$$

on $M(m+1, n+1)$.

(4) If there exist rational numbers $r_1, r_2, \ldots, r_m \in \mathbf{Q}$ such that

$$\sum_{i=1}^{m} r_i(1-c_i) = 1,$$

then the ring

$$R := \mathbf{Q}[c_1^{\pm 1}, c_2^{\pm 1}, \ldots, c_m^{\pm 1}]$$

satisfies the assumption of Theorem 3.3.2.

Example 3.3.4: We give a simple example of Remark 3.3.3,(4); if

$$\alpha_0 = -\frac{m+1}{2}, \qquad \alpha_i = \frac{i}{m} \quad (i = 1, 2, \ldots, m).$$

then we have

$$R = \mathbf{Q}[\exp(\frac{2\pi\sqrt{-1}}{m})].$$

(3.4) Concluding remarks.

Recall that the HGF of type $(m+1, 2)$ is Lauricella's classical HGF F_D. So Theorem 3.3.2 implies that, roughly speaking, *the HGF of type* $(m+1, n+1)$ *restricted to the Segre image is the* n*-th "exterior product" of the Lauricella* F_D:

$$\boxed{\mathrm{HGF}(m+1, n+1)|_{\mathrm{Segre}} = \overset{n}{\wedge} F_D}$$

I am not going to explain what this means exactly, because I do not have enough time.

Anyway, the properties of the Lauricella F_D have been known extensively. So we can say that our HGF's are *known* on the Segre image. Let us draw the following picture (see Figure 1).

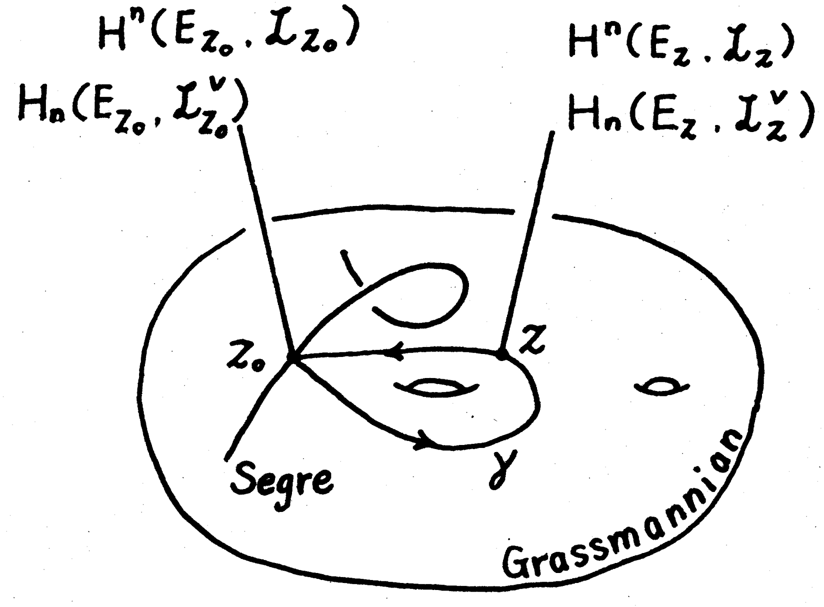

Figure 1.

In order to know the global behaviour of the HGF's, we have to find their monodromy groups. To do so, it is convenient to take a point on the Segre image as a base point of the fundamental groups. Finding the monodromy has been made by K. Matsumoto, T. Sasaki, N. Takayama, M. Yoshida [MSTY] and others.

I would like to stop my talk here. Thank you very much.

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